Computers and Mathematics with Applications 56 (2008) 3188–3194

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Magnetohydrodynamic (MHD) flow of a micropolar fluid towards a stagnation point on a vertical surface

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ARTICLE INFO

Article history:
Received 16 March 2008
Received in revised form 20 July 2008
Accepted 10 September 2008

Keywords:
Dual solutions
Heat transfer
MHD
Micropolar fluid
Stagnation flow

ABSTRACT

The steady MHD mixed convection stagnation point flow towards a vertical surface immersed in an incompressible micropolar fluid is investigated. The external velocity impinges normal to the wall and the wall temperature is assumed to vary linearly with the distance from the stagnation point. The governing partial differential equations are transformed into a system of ordinary differential equations, which is then solved numerically by a finite-difference method. The features of the flow and heat transfer characteristics for different values of the governing parameters are analyzed and discussed. Both assisting and opposing flows are considered. It is found that dual solutions exist for the assisting flow, besides that usually reported in the literature for the opposing flow.

1. Introduction

The theory of micropolar fluids, as developed by Eringen [1,2] has been a field of very active research for the last few decades as this class of fluids represents, mathematically, many industrially important fluids such as paints, body fluids, polymers, colloidal fluids, suspension fluids, etc. These fluids display the effects of local rotary inertia and couple stress and may form suitable non-Newtonian fluid models, which can be used to analyze the behavior of exotic lubricants, animal blood, etc. The study of micropolar fluid mechanics has received the attention of many researchers. A good list of references on the published papers for this fluid can be found in Eringen [3] and Ishak et al. [4]. However, the associated MHD problems have not received that much attention until recently.

The present paper considers a steady MHD flow towards a stagnation point on a vertical surface immersed in a micropolar fluid. The similar problems with non-magnetic effects were considered by Ramachandran et al. [5], Devi et al. [6] and Lok et al. [7]. Ramachandran et al. [5] studied the laminar mixed convection in two-dimensional stagnation flows around heated surfaces by considering both cases of an arbitrary wall temperature and arbitrary surface heat flux variations. They found that a reversed flow developed in the buoyancy opposing flow region, and dual solutions are found to exist for a certain range of the buoyancy parameter. This problem was then extended by Devi et al. [6] to the unsteady case, and by Lok et al. [7] to a surface immersed in a micropolar fluid. As reported by Ramachandran et al. [5], they also found that dual solutions exist only in the opposing flow case. Therefore, the objective of the present study is to show that dual solutions exist for the assisting flow, and not only for the opposing flow as reported previously.

2. Analysis

Consider the steady, two-dimensional flow of an incompressible electrically conducting micropolar fluid near the stagnation point on a vertical heated plate, as shown in Fig. 1. It is assumed that the velocity of the flow external to the

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doi:10.1016/j.camwa.2008.09.013

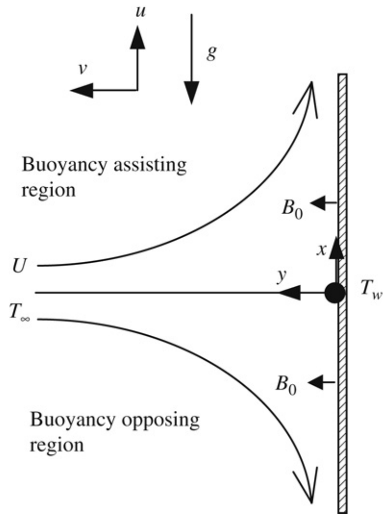

Fig. 1. Physical model and coordinate system.

boundary layer $U(x)$ and the temperature $T_w(x)$ of the plate are proportional to the distance x from the stagnation point, i.e. $U(x) = ax$ and $T_w(x) = T_\infty + bx$, where a and b are constants. A uniform magnetic field of strength B_0 is assumed to be applied in the positive y-direction normal to the plate. The magnetic Reynolds number of the flow is taken to be small enough so that the induced magnetic field is negligible. Under these assumptions along with the Boussinesq and boundary layer approximations, the system of equations, which models the flow is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \left(\frac{\mu+\kappa}{\rho}\right)\frac{\partial^2 u}{\partial y^2} + \frac{\kappa}{\rho}\frac{\partial N}{\partial y} + \frac{\sigma B_0^2}{\rho}(U-u) + g\beta(T-T_\infty), \tag{2}$$

$$\rho j\left(u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y}\right) = \gamma\frac{\partial^2 N}{\partial y^2} - \kappa\left(2N + \frac{\partial u}{\partial y}\right), \tag{3}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{4}$$

subject to the boundary conditions

$$\begin{aligned} &u = 0, \qquad v = 0, \qquad N = -\frac{1}{2}\frac{\partial u}{\partial y}, \qquad T = T_w(x) \quad \text{at } y = 0, \\ &u \to U(x), \qquad N \to 0, \qquad T \to T_\infty \quad \text{as } y \to \infty, \end{aligned} \tag{5}$$

where u and v are the velocity components along the x- and y-axes, respectively, g is acceleration due to gravity and T is the fluid temperature in the boundary layer. Further, $\mu, \kappa, \rho, \beta, j, N, \gamma$ and α are respectively the dynamic viscosity, vortex viscosity (or the microrotation viscosity), fluid density, thermal expansion coefficient, microinertia density, microrotation vector (or angular velocity), spin gradient viscosity and thermal diffusivity. We follow the work of many authors by assuming that $\gamma = (\mu + \kappa/2)j = \mu(1 + K/2)j$, where $K = \kappa/\mu$ is the material parameter. This assumption is invoked to allow the field of equations predicts the correct behavior in the limiting case when the microstructure effects become negligible and the total spin N reduces to the angular velocity (see Ahmadi [8] or Yücel [9]).

The continuity equation (1) is satisfied by introducing a stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{6}$$

The momentum, angular momentum and energy equations can be transformed into the corresponding ordinary differential equations by the following substitutions:

$$\begin{aligned} &\eta = \left(\frac{a}{\nu}\right)^{1/2} y, \qquad f(\eta) = \frac{\psi}{(a\nu)^{1/2}x}, \\ &h(\eta) = \frac{N}{a(a/\nu)^{1/2}x}, \qquad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \end{aligned} \tag{7}$$

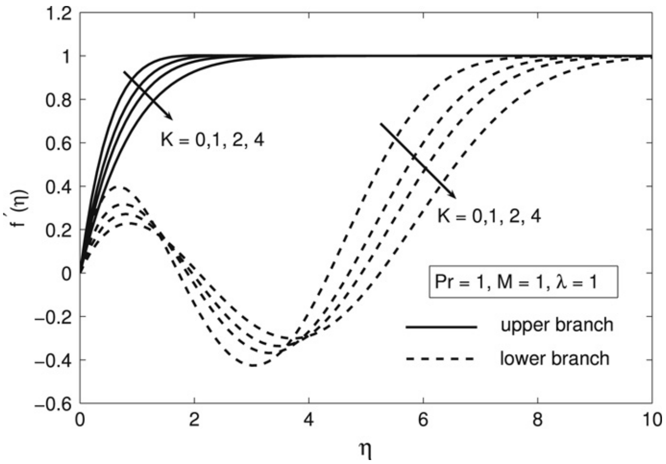

Fig. 2. Velocity profiles $f'(\eta)$ for different values of K when $Pr = 1$, $M = 1$ and $\lambda = 1$ (assisting flow).

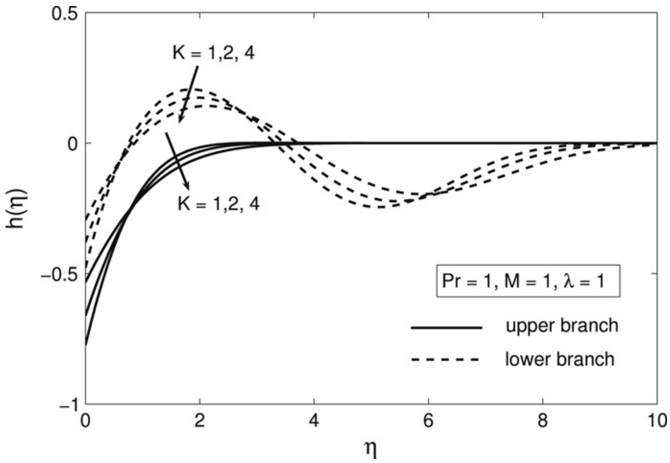

Fig. 3. Angular velocity profiles $h(\eta)$ for different values of K when $Pr = 1$, $M = 1$ and $\lambda = 1$ (assisting flow).

The transformed ordinary differential equations are:

$$(1+K)f''' + ff'' + 1 - f'^2 + Kh' + M(1-f') + \lambda\theta = 0, \tag{8}$$

$$\left(1+\frac{K}{2}\right)h'' + fh' - f'h - K(2h + f'') = 0, \tag{9}$$

$$\frac{1}{Pr}\theta'' + f\theta' - f'\theta = 0, \tag{10}$$

subject to the boundary conditions (5) which become

$$\begin{aligned} &f(0) = 0, \quad f'(0) = 0, \quad h(0) = -\frac{1}{2}f''(0), \quad \theta(0) = 1, \\ &f'(\eta) \to 1, \quad h(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as } \eta \to \infty, \end{aligned} \tag{11}$$

where we have taken $j = \nu/a$ as a characteristic length (see Rees and Bassom [10]). In the above equations, primes denote differentiation with respect to η, $Pr = \nu/\alpha$ is the Prandtl number, $M = B_0^2\sigma/(\rho a)$ is the magnetic parameter and $\lambda = Gr_x/Re_x^2$ is the buoyancy or mixed convection parameter. Further, $Gr_x = g\beta(T_w - T_\infty)x^3/\nu^2$ is the local Grashof number and $Re_x = Ux/\nu$ is the local Reynolds number. We notice that λ is a constant with $\lambda < 0$ and $\lambda > 0$ correspond to the opposing and assisting flows, respectively, while $\lambda = 0$ (i.e. $T_w = T_\infty$) is for pure forced convection flow. When $M = 0$, the problem reduces to those considered by Lok et al. [7], while when $M = 0$ and $K = 0$ it reduces to those of Ramachandran et al. [5].

Table 1
Values of $f''(0)$ for different values of Pr when $M = 0, K = 0$ and $\lambda = 1$

Pr	Ramachandran et al. [5]	Lok et al. [7]	Present results	
			Upper branch	Lower branch
0.7	1.7063	1.706376	1.7063	1.2387
1	–	–	1.6755	1.1332
7	1.5179	1.517952	1.5179	0.5824
10	–	–	1.4928	0.4958

Table 2
Values of $-\theta'(0)$ for different values of Pr when $M = 0, K = 0$ and $\lambda = 1$

Pr	Ramachandran et al. [5]	Lok et al. [7]	Present results	
			Upper branch	Lower branch
0.7	0.7641	0.764087	0.7641	1.0226
1	–	–	0.8708	1.1691
7	1.7224	1.722775	1.7225	2.2191
10	–	–	1.9448	2.4940

The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x, which are defined as

$$C_f = \frac{\tau_w}{\rho U^2/2}, \qquad Nu_x = \frac{x q_w}{k(T_w - T_\infty)}, \tag{12}$$

where the wall shear stress τ_w and the heat flux q_w are given by

$$\tau_w = \left[(\mu + \kappa)\frac{\partial u}{\partial y} + \kappa N\right]_{y=0}, \qquad q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{13}$$

with k being the thermal conductivity. Using the similarity variables (7), we obtain

$$\frac{1}{2} C_f Re_x^{1/2} = \left(1 + \frac{K}{2}\right) f''(0), \qquad Nu_x/Re_x^{1/2} = -\theta'(0). \tag{14}$$

3. Results and discussion

The step size $\Delta\eta$ in η, and the position of the edge of the boundary layer η_∞ have to be adjusted for different values of parameters to maintain accuracy. In this study, the values of $\Delta\eta$ between 0.001 and 0.1 were used, depends on the values of the parameters used, in order that the numerical values obtained are independent of $\Delta\eta$ chosen, at least to four decimal places. However, a uniform grid of $\Delta\eta = 0.01$ was found to be satisfactory for a convergence criterion of 10^{-5} which gives accuracy to four decimal places, in nearly all cases. On the other hand, the boundary layer thickness η_∞ between 6 and 1000 was chosen where the infinity boundary condition is achieved. For some values of the magnetic parameter M, material parameter K and Prandtl number Pr, there is a possibility that two values of η_∞ are obtained for one value of the buoyancy parameter λ, which gives two different velocity, angular velocity and temperature profiles as shown in Figs. 2–7, and in consequence produces two different values of the surface shear stress and the heat transfer rate at the surface, as presented in Figs. 8 and 9, respectively. To assess the accuracy of the present method, comparisons with previously reported data available in the open literature are made for several values of Pr when $K = 0, M = 0$ and $\lambda = 1$, as given in Tables 1 and 2, which show an excellent agreement.

The variations of the skin friction coefficient $f''(0)$ and the local Nusselt number $-\theta'(0)$ with buoyancy parameter λ for $K = 0$ and $K = 1$ are shown in Figs. 8 and 9, respectively, for $Pr = 1$ and $M = 1$. These figures show that it is possible to obtain dual solutions of the similarity Eqs. (8)–(11) also for the assisting flow ($\lambda > 0$), apart from those for the opposing flow ($\lambda < 0$), that have been reported by Ramachandran et al. [5], Devi et al. [6] and Lok et al. [7]. For $\lambda > 0$, there is a favorable pressure gradient due to the buoyancy forces, which results in the flow being accelerated and consequently there is a larger skin friction coefficient than in the non-buoyant case ($\lambda = 0$). For negative values of λ, there is a critical value λ_c, with two branches of solutions for $\lambda > \lambda_c$, a saddle-node bifurcation at $\lambda = \lambda_c$ and no solution for $\lambda < \lambda_c$. The boundary layer separates from the surface at $\lambda = \lambda_c$, thus we are unable to get the solution beyond this value. Based on our computations, $\lambda_c = -3.85$ and $\lambda_c = -4.14$ for $K = 0$ and $K = 1$, respectively, all for $Pr = 1$ and $M = 1$.

The boundary layer separation occurs at $\lambda = \lambda_c$ where $f''(0) < 0$, a different result from the classical boundary layer theory where separation occurs when $f''(0) = 0$. This observation is in agreement with the results reported by Ramachandran et al. [5], Devi et al. [6] and Lok et al. [7]. We identify the upper and lower branch solutions in the following discussion by how they appear in Fig. 8, i.e. the upper branch solution has a higher value of $f''(0)$ for a given λ than the

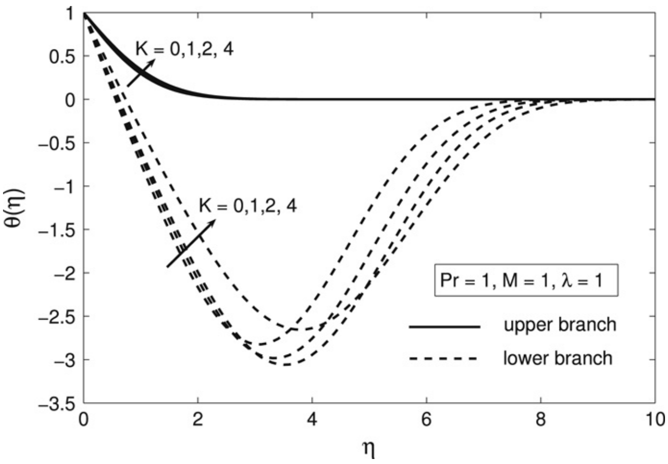

Fig. 4. Temperature profiles $\theta(\eta)$ for different values of K when $Pr = 1, M = 1$ and $\lambda = 1$ (assisting flow).

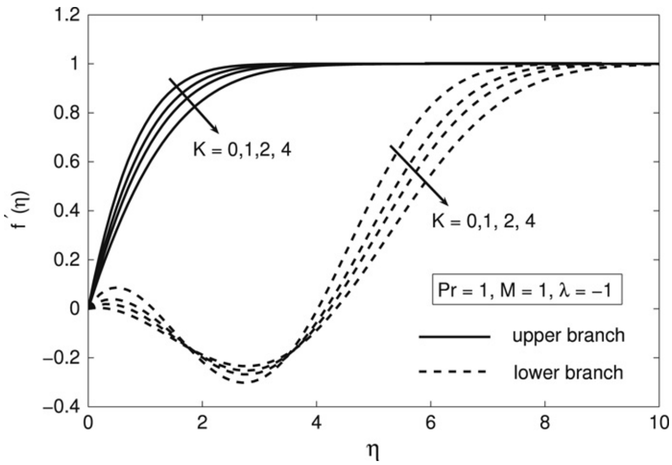

Fig. 5. Velocity profiles $f'(\eta)$ for different values of K when $Pr = 1, M = 1$ and $\lambda = -1$ (opposing flow).

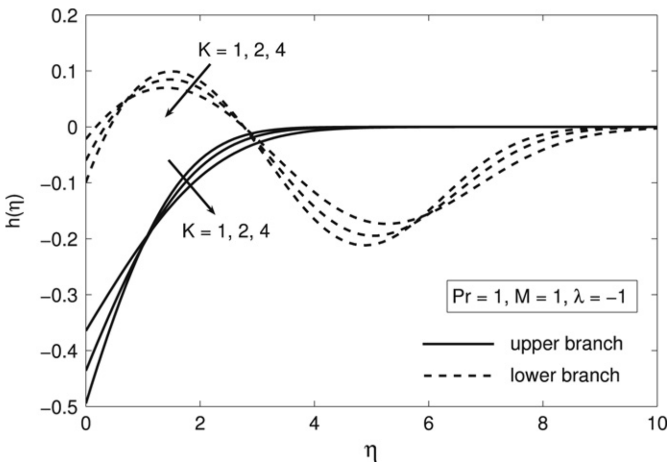

Fig. 6. Angular velocity profiles $h(\eta)$ for different values of K when $Pr = 1, M = 1$ and $\lambda = -1$ (opposing flow).

lower branch solution. For the assisting flow, dual solutions are found to exist for all positive values of λ considered, to much higher values than shown in Fig. 8. This figure also shows that the critical value $|\lambda_c|$ increases as the material parameter K is increased, suggesting that micropolar fluid increases the range of existence of solutions to Eqs. (8)–(11), i.e. micropolar fluids

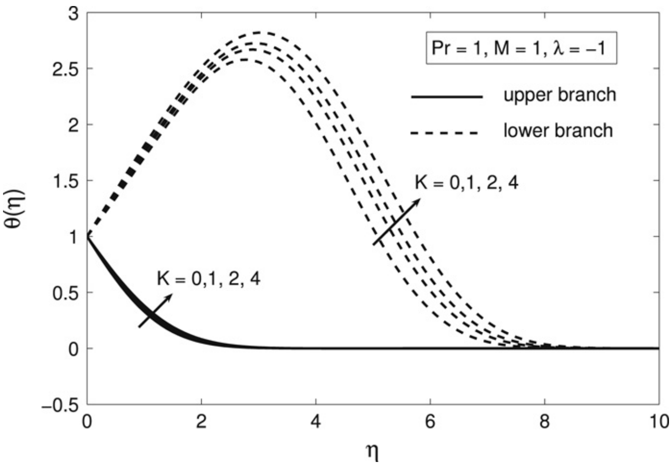

Fig. 7. Temperature profiles $\theta(\eta)$ for different values of K when $Pr = 1$, $M = 1$ and $\lambda = -1$ (opposing flow).

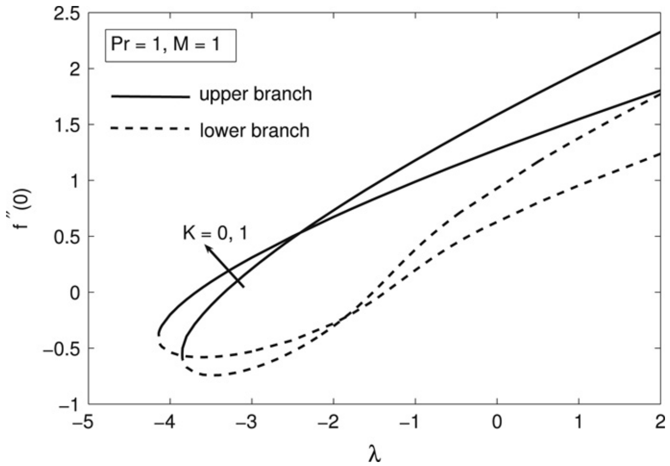

Fig. 8. Variation of the skin friction coefficient $f''(0)$ with λ for $K = 0$ and $K = 1$ when $Pr = 1$ and $M = 1$.

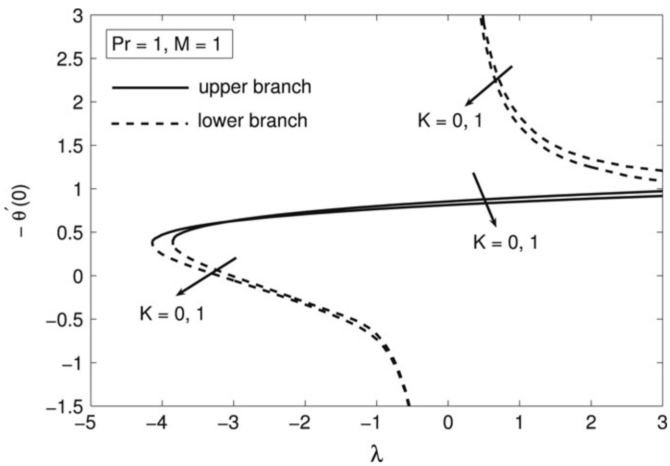

Fig. 9. Variation of the local Nusselt number $-\theta'(0)$ with λ for $K = 0$ and $K = 1$ when $Pr = 1$ and $M = 1$.

delay the boundary layer separation compared to Newtonian fluids. The results shown in Fig. 9 for the heat transfer rate at the surface $-\theta'(0)$ suggest that for the lower branch solution, $-\theta'(0)$ becomes unbounded as $\lambda \to 0^-$ and as $\lambda \to 0^+$.

It is not possible to determine which solution would occur in practice since a stability analysis has not been carried out. However, we expect that the upper branch solution to be stable and physically relevant, whereas the lower branch solution is unstable and not physically relevant, since the upper branch solution is the only solution for the forced convection limit case, $\lambda = 0$ (see Fig. 9). The saddle-node bifurcation at $\lambda = \lambda_c$ corresponds to a change in the (temporal) stability of the solution and, unless there is a change in stability on the upper branch for $\lambda \neq \lambda_c$, the saddle-node bifurcation gives a change in stability from stable (upper branch) to unstable (lower branch). Although the lower branch solutions seem to deprive of physical significance, they are nevertheless of interest so far as the differential equations are concerned. Similar results may arise in other situations where the corresponding solutions have more realistic meaning (see Ridha [11,12]).

4. Conclusions

We have theoretically studied the similarity solutions for the steady MHD flow towards a stagnation point on a vertical surface immersed in an incompressible micropolar fluid. The transformed non-linear ordinary differential equations were solved numerically using an implicit finite difference method. A new feature to emerge from our results is the existence of a reversed flow region, in addition to a dual-solution in the assisting flow regime ($\lambda > 0$). Previous works with non-magnetic effect in viscous fluids [5,6] and micropolar fluids [7] had failed to notice that the lower branch solution could be continued into the assisting flow regime. In the assisting flow case, solutions could be obtained for all positive values of λ, while in the opposing flow case the solution terminated with a saddle-node bifurcation at $\lambda = \lambda_c$ ($\lambda_c < 0$). The value of $|\lambda_c|$ increases with an increase in K, thus micropolar fluid delays the boundary layer separation, which in turn increases the range of similarity solutions compared to Newtonian fluid.

Acknowledgements

The authors wish to express their very sincere thanks to the reviewers for their time spent reading this paper and for their valuable comments and suggestions. This work is supported by research grants, i.e. SAGA fund (STGL-013-2006) from the Academy of Sciences Malaysia and FRGS grant from the Ministry of Higher Education, Malaysia.

References

[1] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966) 1–18.
[2] A.C. Eringen, Theory of thermomicropolar fluids, J. Math. Anal. Appl. 38 (1972) 480–496.
[3] A.C. Eringen, Microcontinuum Field Theories. II: Fluent Media, Springer, New York, 2001.
[4] A. Ishak, R. Nazar, I. Pop, Moving wedge and flat plate in a micropolar fluid, Int. J. Eng. Sci. 44 (2006) 1225–1236.
[5] N. Ramachandran, T.S. Chen, B.F. Armaly, Mixed convection in stagnation flows adjacent to vertical surfaces, ASME J. Heat Transfer 110 (1988) 373–377.
[6] C.D.S. Devi, H.S. Takhar, G. Nath, Unsteady mixed convection flow in stagnation region adjacent to a vertical surface, Heat Mass Transfer 26 (1991) 71–79.
[7] Y.Y. Lok, N. Amin, D. Campean, I. Pop, Steady mixed convection flow of a micropolar fluid near the stagnation point on a vertical surface, Int. J. Numerical Methods Heat Fluid Flow 15 (2005) 654–670.
[8] G. Ahmadi, Self-similar solution of incompressible micropolar boundary layer flow over a semi-infinite flat plate, Int. J. Eng. Sci. 14 (1976) 639–646.
[9] A. Yücel, Mixed convection in micropolar fluid flow over a horizontal plate with surface mass transfer, Int. J. Eng. Sci. 27 (1989) 1593–1602.
[10] D.A.S. Rees, A.P. Bassom, The Blasius boundary-layer flow of a micropolar fluid, Int. J. Eng. Sci. 34 (1996) 113–124.
[11] A. Ridha, Aiding flows non-unique similarity solutions of mixed-convection boundary-layer equations, J. Appl. Math. Phys. (ZAMP) 47 (1996) 341–352.
[12] A. Ridha, Three-dimensional mixed convection laminar boundary-layer near a plane of symmetry, Int. J. Eng. Sci. 34 (1996) 659–675.